Mathematics 272 Lecture 4 Notes

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January 30, 2025

1 Determining Permutons From 4-Point Permutation Densities

1.1 5-point permutation densities determine the uniform distribution

Last time, we wanted to prove the following theorem.

Theorem 1.1. *If $(\pi_n)_{n\in\mathbb{N}}$ is a sequence of permutations $|\pi_n| \to \infty$ and $\lim_{n\to\infty} d(\sigma, \pi_m) = \frac{1}{24}$ for all $\sigma \in S_4$, then $(\pi_n)_{n\in\mathbb{N}}$ and for every permutation σ, $\lim_{n\to\infty} d(\sigma, \pi_n) = \frac{1}{|\sigma|!}$.*

We will first prove a weaker statement.

Theorem 1.2 (Hoeffding, 1948)**.** *If μ is a permuton and $d(\sigma, \mu) = \frac{1}{120}$ for all $\sigma \in S_5$, then μ is the uniform measure.*

Note that this implies that the density of any 3-point permutation in μ is 1/6, and the density of any 4-point permutation in μ is 1/24.

We considered the joint CDF $F(x, y) = \mu([0, x] \times [0, y])$, and we looked at how to calculate the integral $\int F(x, y)\, d\lambda$, where λ is the uniform distribution on $[0, 1]^2$. We showed that

$$\int F(x, y)\, d\lambda = \sum_{\sigma \in S_3} \alpha_\sigma d(\sigma, \mu)$$

for some values α_σ depending only on σ and not on μ.

Proof. Because F is a continuous function (as μ has uniform marginals), it is enough just to look at the integral

$$\int (F(x, y) - xy)^2\, d\lambda = \int F(x, y)^2\, d\lambda - 2F(x, y)xy\, d\lambda + \int x^2y^2\, d\lambda.$$

If we sample $(X_0, Y_0) \sim \lambda$, $(X_1, Y_1) \sim \mu$, and $(X_2, Y_2) \sim \mu$, then

$$\int F(x, y)^2\, d\lambda = \mathbb{P}(X_1 \leq X_0, X_2 \leq X_0, Y_1 \leq Y_0, Y_2 \leq Y_0).$$

If we instead sample $(X_0, Y_0') \sim \mu$ and $(X_0', Y_0) \sim \mu$, then this can be expressed in the exact same way:

$$= \mathbb{P}(X_1 \leq X_0, X_2 \leq X_0, Y_1 \leq Y_0, Y_2 \leq Y_0)$$

Now we just have 4 points according to μ, and we want to look at their relative positions in the square $[0,1]^2$.

$$= \sum_{\sigma \in S_4} \alpha_\sigma d(\sigma, \mu)$$

For example, $\alpha_{1\ 2\ 3\ 4} = \frac{4}{24}$.

Similarly,

$$\int F(x,y)xy\, d\lambda = \mathbb{P}(X_1 \leq X_0, Y_1 \leq Y_0, X_2 \leq X_0, Y_3 \leq Y_0),$$

where $(X_0, Y_0) \sim \lambda$, $(X_1, Y_1) \sim \mu$, $(X_2, Y_2) \sim \lambda$, and $(X_3, Y_3) \sim \lambda$. We can actually take $(X_2, Y_2) \sim \mu$, $(X_3, Y_3) \sim \mu$ because μ has uniform marginals. And, by instead sampling $(X_0, Y_0') \sim \mu$ and $(X_0', Y_0) \sim \mu$, we can get

$$= \sum_{\sigma \in S_5} \alpha_\sigma d(\sigma, \mu)$$

for some coefficients α_σ.

The third term is a constant, independent of μ. We see that the right hand side is a value which doesn't depend on μ (as long as μ has the same permutation densities as λ), and since the left hand side equals 0 when μ is the uniform distribution, this must equal 0 in general. So $\mu = \lambda$, as claimed. □

1.2 4-point permutation densities determine the uniform distribution

We would now like to prove a version of the theorem with 4-point permutations, rather than 5-point permutations.

Theorem 1.3. *If μ is a permuton and $d(\sigma, \mu) = \frac{1}{24}$ for all $\sigma \in S_4$, then μ is the uniform measure.*

Proof. A similar argument from before tells us that

$$\left(\int F(x,y)xy\, d\mu \right)^2 = \sum_{\sigma \in S_4} \alpha_\sigma d(\sigma, \mu)$$

for some constants α_σ. So this is the same as if we had $\mu = \lambda$, which gives us the value

$$= \frac{1}{81}.$$

On the other hand, using Cauchy-Schwarz,

$$\left(\int F(x,y)xy\,d\mu\right)^2 \le \left(\int F(x,y)^2\,d\mu\right)\left(\int x^2y^2\,d\mu\right)$$

The first term can be expressed as $\sum_{\sigma\in S_3}\alpha'_\sigma d(\sigma,\mu)$ for some constants α'_σ, so it equals the same value as if $\mu=\lambda$. In particular, we get

$$=\frac{1}{9}\int x^2y^2\,d\mu.$$

The second term is the hard part, since it still requires us to sample 5 points from μ.

It will be helpful for us to understand $\int F(x,y)xy\,d\lambda$.We can rearrange this by writing it as two integrals and swapping the order.

$$\begin{aligned}\int F(x,y)xy\,d\lambda &= \int \mathbb{1}_{\{x'\le x,y'\le y\}}xy\,d\mu(x',y')\,d\lambda(x,y)\\ &= \iint \mathbb{1}_{\{x'\le x,y'\le y\}}xy\,d\lambda(x,y)\,d\mu(x',y')\\ &= \int \frac{1}{2}(1-(x')^2)\frac{1}{2}(1-(y')^2)\,d\mu\\ &= \frac{1}{4}\int 1-x^2-y^2+x^2y^2\,d\mu(x,y)\end{aligned}$$

. Substituting this into the inequality we had before, we get

$$\begin{aligned}\frac{1}{81} &= \frac{1}{9}\int x^2y^2\,d\mu\\ &\le \frac{1}{9}\left(4\int F(x,y)xy\,d\lambda - \int 1-x^2-y^2\,d\mu\right)\\ &= \frac{1}{9}\left(4\int F(x,y)xy\,d\lambda - \frac{1}{3}\right)\end{aligned}$$

Using Cauchy-Schwarz again,

$$\begin{aligned}&\le \frac{4}{9}\sqrt{\int F(x,y)^2\,d\lambda}\sqrt{\int x^2y^2\,d\lambda}-\frac{1}{27}\\ &= \frac{4}{9\cdot 3\cdot 3}-\frac{1}{27}\\ &= \frac{1}{81}.\end{aligned}$$

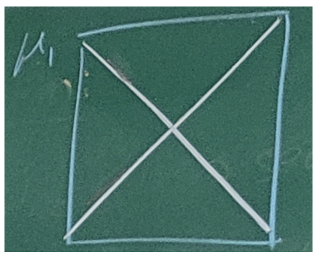
Therefore, all the inequalities are actually equalities, and we get $F(x,y)=c\cdot xy$ almost everywhere for some constant c. So we get that $\mu=\lambda$, as desired. □

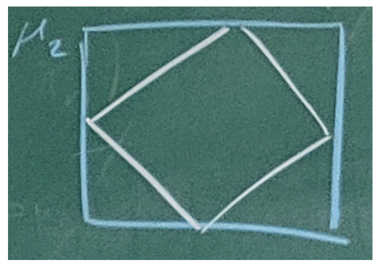
1.3 Proving convergence of permutations from convergence of 4-point permutation densities

To finish the proof of Theorem 1.1, we need to show that (π_n) is convergent.

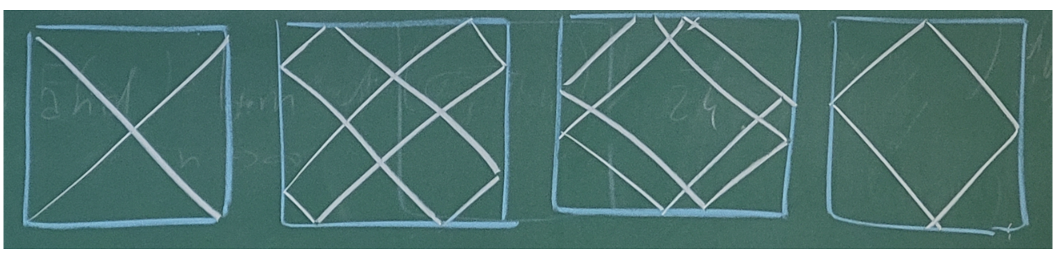
Proof. Suppose $(\pi_n)_{n\in\mathbb{N}}$ is such that $|\pi_n| \to \infty$ and $\lim_{n\to\infty} d(\sigma, \pi_n) = \frac{1}{24}$. If $(\pi_n)_{n\in\mathbb{N}}$ is not convergent, then there exist two subsequences $(n_i)_{i\in\mathbb{N}}, (n'_i)_{i\in\mathbb{N}}$ and a permutation σ such that $\lim_{i\to\infty} d(\sigma, \pi_{n_i}) \neq \lim_{i\to\infty} d(\sigma, \pi_{n'_i})$. Without loss of generality, we may assume that $(\pi_{n_i})_{i\in\mathbb{N}}$ are convergent (by compactness) with limits μ, μ'. However, our theorem says that both μ, μ' are the uniform measure, which implies that $d(\sigma, \mu) = d(\sigma, \mu')$. □

Example 1.1. Here is an example that shows that we cannot prove the same theorem with S_3 in place of S_4. Define

$\mu_1 =$ 

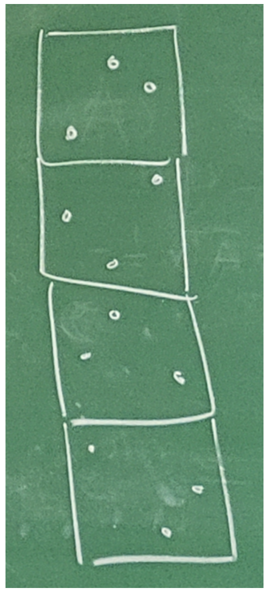
$\mu_2 =$ 

Then define $\mu_\alpha = \alpha\mu_1 + (1-\alpha)\mu_2$, interpolating between μ_1, μ_2.



Then $d(\ 1\ \ 2\ \ 3\ , \mu_\alpha)$ is a continuous function in α, so there exists some α which gives us $d(\ 1\ \ 2\ \ 3\ , \mu_\alpha) = 1/6$. By symmetry, $d(\ 3\ \ 2\ \ 1\ , \mu_\alpha) = 1/6$. On the other hand, by symmetry, we have

$$d(\ 1\ \ 3\ \ 2\ , \mu_\alpha) = d(\ 2\ \ 1\ \ 3\ , \mu_\alpha) = d(\ 2\ \ 3\ \ 1\ , \mu_\alpha) = d(\ 3\ \ 1\ \ 2\ , \mu_\alpha),$$



so these all must equal $\frac{1}{4}(1 - 2\frac{1}{6}) = \frac{1}{6}$.

Here are two theorems that a computer can prove using the flag algebra method we will see later.

Theorem 1.4. *If μ is a permuton such that $d(1\ 2\ 3, \mu) = d(3\ 2\ 1, \mu) = \frac{1}{6}$ and $d(2\ 1\ 4\ 3, \mu) = d(3\ 4\ 1\ 2, \mu) = d(2\ 4\ 1\ 3, \mu) = d(3\ 1\ 4\ 2, \mu) = \frac{1}{24}$, then μ is the uniform measure.*

Theorem 1.5. *If μ is a permuton such that*

$$d(1\ 2\ 3\ 4, \mu) + d(2\ 1\ 3\ 4, \mu) + d(2\ 1\ 4\ 3, \mu) + d(1\ 2\ 4\ 3, \mu)$$
$$+ d(4\ 3\ 2\ 1, \mu) + d(4\ 3\ 1\ 2, \mu) + d(3\ 4\ 1\ 2, \mu) + d(3\ 4\ 2\ 1, \mu) = \frac{1}{3},$$

then μ is uniform.